



Selfish bin covering[☆]

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ABSTRACT

In this paper, we address the selfish bin covering problem, which is greatly related both to the bin covering problem, and to the weighted majority game. What we are mainly concerned with is how much the lack of central coordination harms social welfare. Besides the standard PoA and PoS, which are based on Nash equilibrium, we also take into account the strong Nash equilibrium, and several new equilibrium concepts. For each equilibrium concept, the corresponding PoA and PoS are given, and the problems of computing an arbitrary equilibrium, as well as approximating the best one, are also considered.

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1. Introduction

The bin covering problem is also called the *dual bin packing problem*, because it has a dual relation with the famous bin packing problem, which can be observed immediately from the definition below. It has $n + 1$ positive integers, a_1, a_2, \dots, a_n, b , as input, where a_j ($1 \leq j \leq n$) is the *size* of item j and b the *volume* of bins. We assume that $a_j < b$ for all $1 \leq j \leq n$ and there are sufficiently many bins. The problem is to allocate the items into bins such that the number of *covered* bins is maximized, where a bin is called covered iff its *load*, i.e. the total size of items allocated to it, is equal to or greater than b . Alternatively, the problem can be seen as to partition the n items such that the number of subsets in the partition whose loads are not less than b is maximized.

This simple abstract model may have various interpretations in the real world. The item may be just a lifeless object, say a stamp. It may also be a person or a company. When the item is interpreted as a lifeless object, its size can be seen as either simply the size, or the weight, or the value. When it is interpreted as a person/company, its size can be treated as the amount of a certain kind of resource it possesses, say money/market share. A group of items “cooperate” to do a task by pooling their resources together, and the volume of bins is interpreted as the threshold for success. Therefore the whole story can be told either as partitioning stamps into as many as possible groups such that the total value of each group covers the cost of mailing an envelop, or as partitioning people into as many as possible groups such that each group of people has enough money for running a company, or as partitioning companies into as many as possible groups such that each group has a big enough total market share.

An implicit assumption in the bin covering problem is the existence of a central decision maker, who has absolute control of all the items. This kind of centralized system is often encountered in the real world, and has its own advantages, say, efficiency. At the same time, centralized systems are usually hard to realize and fragile. When items are people or companies,

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Table 1
Main results

Equilibria	PoA	PoS	Complexity	Approximability
NE	0	1	$O(1)$	AFPTAS
FNE(I)	0.5	1	$O(n^2)$	AFPTAS
FNE(II)	0.5	1	NP-hard	NP-hard
FNE(III)	0.5	1	NP-hard	NP-hard
M-SNE	0.5	1	NP-hard	NP-hard
SNE	0.5	0.5	NP-hard	NP-hard

it is likely that they are reluctant to be in the charge of the central decision maker for various reasons, and lie about their real sizes, which are usually their private information. And once the decision maker makes a mistake, it might be fatal to the whole system.

There is also an opposite kind of system, the decentralized system, where the central decision maker disappears, and is replaced by a set of agents, each of which controls a single item. When items are interpreted as people or companies, agents are simply items. For simplicity, we shall not distinguish between an item and the corresponding agent in the sequel, and use both terms.

Compared with the centralized system, the decentralized system is usually easier to realize and more robust. The biggest disadvantage of decentralized systems, however, is that in most cases, the lack of a central decision maker harms the welfare of the system, which is referred to as the social welfare in the rest of this paper. That is, it is usually inefficient. The natural question is, how inefficient? If the inefficiency is relatively small, then it may not matter.

An immediate relevant question is, how to quantify efficiency?

Armed with game theory, researchers find two satisfactory measures: PoA (Price of Anarchy, [21]) and PoS (Price of Stability, [28]). Game theory studies the scenario where all the players (agents) are self-interested, and takes equilibria as the outcomes that will happen. Since equilibria are usually not unique, PoA is intuitively defined as the ratio of the worst social welfare we may get among all the equilibria in the decentralized system divided by that in the centralized system (i.e. the optimal social welfare), and PoS is defined as the ratio of the best social welfare among all the equilibria divided by the optimal social welfare. The analysis of PoA and PoS for various decentralized systems, forms an important part of the booming *algorithmic game theory* [25,27].

We refer to the decentralized bin covering problem as the *selfish bin covering* problem (SBC for short). Before giving the formal definitions of PoA and PoS for SBC, there are still three questions we have to answer: 1. What are the payoff functions of agents? 2. How to define social welfare? 3. How to define an equilibrium?

For the first question, we take the natural and frequently used proportional rule. That is, in any covered bin, the payoff allocated to any member is its size divided by the load of the bin (we assume that once a group of agents is big enough for success, i.e. the corresponding bin is covered, it will get a fixed profit of 1). For the second, we take the popular *utilitarian* welfare function, i.e. the sum of all the agents' payoff, which is exactly the number of covered bins in our model. For the last, we take the famous (pure) Nash equilibrium (NE for short), (pure) strong Nash equilibrium (SNE), and four newly defined ones: fireable Nash equilibria type (I) (FNE(I)), fireable Nash equilibria type (II) (FNE(II)), fireable Nash equilibria type (III) (FNE(III)), and modified strong Nash equilibrium (M-SNE), whose exact definitions will be given in the later corresponding sections.

Formally, let SBC be the set of instances of the selfish bin covering problem, $OPT(I)$ be the optimal social welfare of $I \in SBC$, $NE(I)$ be the set of NEs of I , and $p(\pi)$ be the social welfare of $\pi \in NE(I)$, then PoA and PoS for SBC w.r.t. NE are defined as follows:

$$PoA^{NE} = \inf_{I \in SBC} \min_{\pi \in NE(I)} \frac{p(\pi)}{OPT(I)},$$

$$PoS^{NE} = \inf_{I \in SBC} \max_{\pi \in NE(I)} \frac{p(\pi)}{OPT(I)}.$$

Analogous to PoA^{NE} and PoS^{NE} , we can define the other PoAs and PoSs. What we need to do is simply to replace NE in the above two formulas with SNE etc., respectively.

For each equilibrium, we are also concerned with the problems of computing an arbitrary equilibrium, as well as approximating the best one (which is NP-hard to find). Our results are listed in Table 1, where “Complexity” stands for the complexity of finding an arbitrary equilibrium, and “AFPTAS” (Asymptotic Fully Polynomial Time Approximation Scheme) means that AFPTAS exists, that is, for arbitrary positive small ϵ , there is an algorithm which outputs an equilibrium whose social welfare is at least $1 - \epsilon$ times the optimal one when the optimal social welfare is large enough, and the running time can be bounded by a polynomial both in n and $1/\epsilon$.

2. Related work

The bin covering problem was first studied by Assman in his thesis [1] and then in the journal paper of Assmann et al. [2]. It is very easy to show that the bin covering problem is NP-hard, and we are not able to approximate it with a worst case

performance ratio better than 0.5, unless $P = NP$ [1]. It is still not hard to show that the next-fit algorithm has a worst case performance ratio of exactly 0.5 [2]. Analogous to the bin packing problem, what we are interested is how the bin covering problem can be asymptotically approximated. To put it precisely, given any instance I , let $A(I)$ be the number of bins covered by algorithm A , and $OPT(I)$ be the optimal value. The *asymptotic worst case ratio* of algorithm A is defined as

$$R_{\infty}^A = \liminf_{OPT(I) \rightarrow \infty} \frac{A(I)}{OPT(I)}.$$

One central topic for the bin covering problem is to find a polynomial time algorithm with an asymptotic worst case ratio as large as possible.

Low-order polynomial time algorithms with $R_{\infty}^A = 2/3$ and $3/4$ are explored by Assman [1], Assmann et al. [2], and Csirik et al. [8]. Csirik et al. [9] give the first APTAS (Asymptotic Polynomial Time Approximation Schemes), Jansen and Solis-Oba [20] derive an AFPTAS. For the analysis of average case performance, there are also several results [7,9]. Woeginger and Zhang [30] also consider the variant with variable-sized bins.

The weighted majority game (WMG for short), which is a classic model in coalitional (cooperative) game theory and has rich applications in politics, is greatly related with the selfish bin covering problem, hence we give a sketch of it. An instance of WMG is usually represented by $G = (q; a_1, a_2, \dots, a_n)$, where q is called the quota and a_j the weight of player j . The characteristic function is defined as: $v(C) = 1$ iff $\sum_{j \in C} a_j \geq q$; $v(C) = 0$ otherwise. q is usually required to be bigger than half of the total weights.

The story is usually told as: there is a fixed amount of money to be allocated among n players and they vote for how to allocate it. Any group of players can cooperate to present a scheme of allocating all the money among them. Each player j has a_j votes, and any scheme getting more than q votes will get through.

Trivially, the core of WMG is usually empty. An interesting phenomenon in WMG is that the bargaining powers of players usually are not proportional to their weights. For a simple example, there are 3 players in total, their weights are 3, 4, 5, respectively, and the quota is 7, then all the three players have the same bargaining power, because any two of them can win the game by cooperation. If we change the first player's number of votes from 3 to 1, then his bargaining power will be 0, because he is absolutely useless. The above two arguments are both very strong and counter-intuitive. Therefore a central topic in WMG is how to measure the bargaining powers of the players. This topic has an obvious significance in political science.

Therefore, WMG and SBC are closely related, the main differences are that: (1) q should be bigger than a half of the total weights, while there is no such restriction for b ; (2) Research of SBC, as well as of classical bin covering, in the field of combinatorial optimization, focuses on optimization and computational aspects (this paper included), and that of WMG, in the fields of game theory and political science, focuses on how to measure bargaining powers.

There are mainly four recognized ways for measuring bargaining powers: the Shapley–Shubik index [29], the Banzhaf index [3], the Holler–Packel index [17,18], and the Deegan–Packel index [11]. These power indices also provide us new insights on how to allocate the profit more fairly in SBC: proportional allocation according to power indices is obviously much fairer than according to weights.

Matsui and Matsui [23,24] prove that all the problems of computing the Shapley–Shubik indices, the Banzhaf indices and the Deegan–Packel indices in WMGs are NP-hard, and there are pseudo-polynomial time dynamic programming algorithms for them. Cao and Yang [10] show that computing the Holler–Packel index is also NP-hard. Deng and Papadimitriou [12] prove that it is #P-complete to compute the Shapley–Shubik index. Matsui and Matsui [24] observe that Deng and Papadimitriou's proof can be easily carried over to the problem of computing the Banzhaf index.

Though the size-proportional way of payoff allocation in our paper may not be absolutely fair, in most cases the proportion does roughly measure the contribution that an agent makes to a coalition. What is more, this way prevails in the real world, and its being trivial to calculate, compared with the NP-hardness of the more delicate ways, is really an advantage to computational analysis.

Coalition formation games, which studies which coalition structures (i.e. partitions of players) are the most likely to form, is a relatively new branch in coalitional game theory. We can safely classify SBC into coalition formation games. In coalition formation games, *stable* coalition structures are treated as the ones to form. Since the concept of stability is just another word for equilibrium, which we discussed in the last section, briefly introducing several closely related stabilities that have already been used in coalition formation games will be helpful.

Nash stability, i.e. NE that we mentioned in the last section, requires that no player should benefit if she leaves the original coalition to which she belongs and joins a new one. In this concept, there is no restriction for the players' migrations, i.e. as long as a player likes to migrate to a coalition, neither the other players nor any other external factor can forbid her doing so. In the real world, however, people rarely have such kind of absolute freedom, especially when their actions harm others. Therefore, it is very reasonable to introduce restrictions on migrations of players. Notice first the subtle fact that the more restriction we place on the players' migrations, the weaker the concept we will get.

The migration of any player may usually affect the benefits of two groups of players, her old colleagues, i.e. members of the coalition to which she belonged originally (herself not accounted), and her colleagues-to-be, i.e. members of the coalition that she is going to join (if externality is considered, other players may also be affected). To require that any player's migration should not harm her colleagues-to-be gives the concept of *individual stability* (IE for short). To require that any player's migration should harm neither her colleagues-to-be nor her old colleagues gives *contractually individual stability* (CIE for short).

Because the restriction for players' migration is the least in NE and the most in CIE, we know that NE implies IE, and IE further implies CIE. For more knowledge about IE and CIE, please refer to [6].

It is also straightforward that if X and Y are two equilibrium concepts, and X is stronger than Y , then

$$PoA^Y \leq PoA^X \leq PoS^X \leq PoS^Y.$$

Since we will show that $PoA^{NE} = 0$ and $PoS^{NE} = 1$, IE and CIE will be omitted in our later discussion.

Selfish bin packing (SBP for short), which combines the idea of decentralization into the classic bin packing problem, is the most related parallel model to SBC. In SBP, there are also n items, each of which has a size and is controlled by a selfish agent, and sufficiently many bins with identical capacities b . The difference is that the load of any bin should not exceed b , and every nonempty bin incurs a cost of 1, which is shared among its members proportional to their sizes.

This model was introduced by Bilò [4]. The exact PoA^{NE} is still unknown up to now, and the current best lower bound and upper bound are 1.6416 (by Epstein and Kleiman [13], Yu and Zhang [31], independently) and 1.6428 (by Epstein and Kleiman [13]), respectively, with a narrow gap to cover. Epstein and Kleiman also show that $PoS^{NE} = 1$, and $1.6067 \leq PoA^{SNE} = PoS^{SNE} \leq 1.6210$. Yu and Zhang show that computing an NE can be done in $O(n^4)$ time.

3. Notations and preliminaries

We denote by N the set of agents, i.e. $N = \{1, 2, \dots, n\}$. Let $\pi = \{B_1, B_2, \dots, B_m\}$ be a partition of N , that is $\bigcup_{i=1}^m B_i = N$, and $B_i \cap B_j = \emptyset$ for all $1 \leq i \neq j \leq m$. We do not distinguish between a bin and the subset of agents in this bin, if no confusion is incurred. For any subset B_i of N , $s(B_i)$ is the load of B_i , i.e. $s(B_i) = \sum_{j \in B_i} a_j$.

A partition π is called *reasonable* if there is at most one B_i ($1 \leq i \leq m$) such that $s(B_i) < b$. Since unreasonable partitions are quite uninteresting, we restrict our discussion to reasonable partitions in this paper.

We use $B_{j,\pi}$ to denote the member in π to which agent j belongs. Let $p(\pi)$ be the social welfare of π , i.e. the number of covered bins in π , $p(j, \pi)$ be the payoff allocated to agent j in π , then $p(j, \pi) = a_j/s(B_{j,\pi})$ if $s(B_{j,\pi}) \geq b$, and $p(j, \pi) = 0$ if $s(B_{j,\pi}) < b$. For any $B_i \in \pi$, let $a_{\min}(B_i) = \min\{a_j : j \in B_i\}$. A bin B_i is called *minimal covered* if it is covered and the removal of any of its member causes it uncovered, i.e. $b \leq s(B_i) < b + a_{\min}(B_i)$, and *exactly covered* if $s(B_i) = b$. π is called *rational* if it is reasonable and all the covered bins are minimal covered.

The partition problem [16], which may be the easiest NP-hard problem, is frequently used in the proving of weakly NP-hardness. It is described as follows: given a set of positive integers e_1, e_2, \dots, e_n , can we partition these integers into two subsets, such that each of them has exactly a half of the total size i.e. is there a set $C \subseteq \{1, 2, \dots, n\}$ such that $\sum_{j \in C} e_j = \sum_{j \notin C} e_j$?

The subset sum problem [16], which will be used later, is also a classic problem. Given a set of positive integers e_1, e_2, \dots, e_n and an integer s , the problem asks if there exists a subset which sums to s . Subset sum is NP-hard, and admits a pseudo-polynomial time algorithm with time complexity $O(n \sum_{j=1}^n e_j)$. The subset sum problem can be seen as a special case of the knapsack problem, which is known to be linearly solvable if the capacity of the knapsack is a constant [26]. Therefore, if s is a constant, the subset sum problem is also linearly solvable.

FFD (First Fit Decreasing, [19]), which is a well known heuristic for the bin packing problem, has an analogue in the bin covering problem. We still denote the analogue as *FFD*. It first sorts the items in non-increasing order of their sizes, then puts the items one by one in this order to a bin until the bin is covered, and opens a new bin to repeat the above action if there are still unpacked items. Due to the non-increasing order, we can easily see that bins derived by *FFD* are all minimal covered, except possibly the last uncovered one.

LPT (Largest Processing Time, [15]), which is a well known heuristic for the parallel machine scheduling problem $P||C_{\max}$, will also be used later. *LPT* first sorts all the jobs in non-increasing order of their processing times, and then allocates the jobs one by one in this order to a least loaded machine. In SBC, the size of an item can be naturally seen as the processing time, and bins can be seen as machines. In the next section, we will use *LPT* partially. That is, a subset of items have already been allocated to bins, and *LPT* is used to allocate the remaining items to the already opened bins.

4. Nash equilibrium

According to the definition, a reasonable partition $\pi = \{B_1, B_2, \dots, B_m\}$ is an NE iff $p(j, \pi(j, B_i)) \leq p(j, \pi)$ for all $j \in N$ and for all B_i such that $j \notin B_i$, where $\pi(j, B_i)$ is a partition of N derived by moving j to B_i . NE always exists, since it is trivial to see that $\{N\} \in NE$. Remember that we assume $a_j < b$ for all $j \in N$. And therefore $PoA^{NE} = 0$.

From any reasonable partition $\pi = \{B_1, B_2, \dots, B_m\}$, we show that it is easy to derive an NE, without decreasing the social welfare. Suppose that the unique uncovered bin in π is B_m .

Algorithm FFD-LPT

STEP 1. Input the reasonable partition $\pi = \{B_1, B_2, \dots, B_m\}$;

for $1 \leq i \leq m - 1$

while (B_i is not minimal covered)

 {take out the smallest remaining item;}

endfor

STEP 2. Run FFD on the taking out items and the items in B_m ;
 /*Without loss of generality, we assume there are still $m - 1$ covered bins in total, and the unique uncovered bin is B_m .*/
 STEP 3. **while** ($\max\{a_j : j \in B_m\} > \min\{a_j : j \in \cup_{i=1}^{m-1} B_i\}$)
 {exchange the biggest item in B_m with the smallest one in $\cup_{i=1}^{m-1} B_i$;
 /*Suppose B_k is the bin which contains the smallest item in $\cup_{i=1}^{m-1} B_i$ before the exchange.*/
 while (B_k is not minimal covered)
 {take out the smallest remaining item;}
 Run FFD on the taking out items and items in B_m .}
 STEP 4. Call LPT to allocate the items of B_m into B_1, B_2, \dots, B_{m-1} .

Lemma 1. For any reasonable partition π , FFD–LPT outputs an NE in $O(n^2)$ time, without decreasing the social welfare.

Proof. Let π' be the final partition. It is trivial that $p(\pi') \geq p(\pi)$. Without loss of generality, we suppose $\pi' = \{B'_1, B'_2, \dots, B'_{m-1}\}$. To verify that $\pi' \in NE$, it suffices to show that for all $1 \leq i \leq m - 1$, the smallest agent in B'_i is satisfied with its position.

We denote the partition at the end of Step 3 still as $\pi = \{B_1, B_2, \dots, B_m\}$, and have the next three observations: (a) each item in $\cup_{i=1}^{m-1} B_i$ is not bigger than any one in B_m ; (b) each B_i ($1 \leq i \leq m - 1$) is minimal covered; (c) every agent in $\cup_{i=1}^{m-1} B_i$ is satisfied with its position in the partial partition $\{B_1, B_2, \dots, B_{m-1}\}$. (a) is just an interpretation of the stopping condition of Step 4, (b) is due to FFD, and (c) comes directly from (b).

For any $1 \leq i \leq m - 1$ and $j \in B'_i \in \pi'$ with $a_j = a_{\min}(B'_i)$, we discuss in two cases. **Case 1.** j is in B_m at the end of Step 3. Due to LPT, B_i has the smallest total size before the joining of j , and therefore j is temporarily satisfied with her position. $a_j = a_{\min}(B_i)$ tells us that j is the last member of B_i . After the joining of j , the total size of B_i keeps unchanged, while the total sizes of the other bins do not decrease, hence j is satisfied with her position in the final partition π' . **Case 2.** j is in B_i , $1 \leq i \leq m - 1$, at the end of Step 3. Due to observation (a), there is no member joining B_i in Step 4. Since agent j is satisfied at the end of Step 3 (observation (c)), she is satisfied in the final partition π' .

Step 1 and Step 2 can be done in $O(n \log n)$ time. Since no exchanged item in B_m comes back, the outer while loop in Step 3 is executed for at most n times. And it is easy to see that each loop can be done in $O(n)$ time. Therefore the total time for Step 3 is $O(n^2)$. Step 4 can be easily done in $O(n \log n)$ time. In sum, the total running time is $O(n^2)$. \square

Theorem 1. (a) $PoA^{NE} = 0$, $PoS^{NE} = 1$;

(b) There exists an AFPTAS for approximating the best NE.

Proof. Since there is always an optimal partition that is reasonable, we immediately have (a) from Lemma 1. For any partition derived by the AFPTAS for the bin covering problem [20], we can also give a corresponding NE by FFD–LPT without decreasing the social welfare, hence (b) is valid. \square

5. Fireable Nash equilibria

Let $\pi = (B_1, B_2, \dots, B_m)$ be a reasonable partition. We assume that B_m is the unique uncovered bin (this will not lose any generality). We shall define three fireable Nash equilibria. The common basic idea behind these equilibrium concepts is that, in any stable partition every covered bin should be minimal covered, because otherwise certain agents will be “fired”. We can either assume that there is a “boss” for each bin who does not like redundancy, or assume that there is “violent democracy” in every bin: any agents will be kicked out as long as the other agents unanimously agree to do so. The process of firing can also be viewed identically as a special kind of group deviation: no agent can be kicked out, but agents from the same bin can leave and form a new coalition. From this point of view, we know that fireable Nash equilibria are weaker than strong Nash equilibrium.

The differences of the three concepts lie in how to constrain the migrations of agents. Recall that in NE, there is no constraint for the agents' migrations; in IE, the migration of an agent should not harm her new partners; and in CIE, neither should the migration harm her new partners nor should it harm her old colleagues.

Remember the definition of rational partitions in Section 3. It says exactly the idea that no redundant bins should appear. Therefore in all the three definitions below, “rational” is a first requirement.

5.1. Definitions

Definition 1. $\pi \in FNE(I)$ iff:

(1) π is rational.

(2) There exists no item $j \in \cup_{i=1}^{m-1} B_i$ such that (a) $B_m \cup \{j\}$ is minimal covered, and (b) $s(B_m) + a_j < s(B_{j,\pi})$.

The two conditions in (2) guarantee that agent j will be strictly better off if she moves to B_m , and $B_m \cup \{j\}$ is stable in the sense that no agent will be kicked out in the future. Therefore when such a j exists, she will be definitely welcome by all agents in B_m , and thus the original partition is not “stable”.

In FNE(I), we require that migration of any agent from a minimal covered bin into the uncovered bin B_m be not allowed, if it causes B_m to be non-minimal covered. This is justified because some agents in B_m worry that they will be eventually fired by the new bin and thus say “no” to the immigration.

Definition 2. $\pi \in \text{FNE(II)}$ iff:

- (1) π is rational.
- (2) There exists no item $j \in \cup_{i=1}^{m-1} B_i$ and a subset E of B_m such that (a) $(B_m \setminus E) \cup \{j\}$ is minimal covered, and (b) $s(B_m \setminus E) + a_j < s(B_{j,\pi})$.

In FNE(II), the migration that we remarked for FNE(I) is allowed, this can be justified by assuming that all agents in B_m are friendly. Since their interests will never be harmed, they say “yes” to any immigration. Another implicit assumption in the above definition is that the migrating agent is farsighted: if she benefits from the migration only after the firing of some subset of the original members, which she anticipates will happen, then she will do it. Note that the migration of any agent, either from a minimal covered bin or from the uncovered bin, into a minimal covered bin, is not allowed, even if the migrating agent anticipates that this will benefit her eventually. Further allowing this kind of migration gives FNE(III).

Definition 3. $\pi \in \text{FNE(III)}$ iff:

- (1) π is rational.
- (2) There exists no item $j \in N$, a bin B_i ($1 \leq i \leq m$, $B_i \neq B_{j,\pi}$) and a subset E of B_i such that (a) $(B_i \setminus E) \cup \{j\}$ is minimal covered, and (b) $s(B_i \setminus E) + a_j < \min\{\delta(B_i), \delta(B_{j,\pi})\}$, where $\delta(B) = s(B)$ if $s(B) \geq b$, and $\delta(B) = +\infty$ otherwise, for any $B \subseteq N$.

In the above definition, it is easy to check that (b) means that if the new bin $(B_i \setminus E) \cup \{j\}$ forms, then all its members will be strictly better off than in π . It is also easy to see that FNE(III) implies FNE(II), which further implies FNE(I), since the constraints for agents' migrations are the most in FNE(I), and are the least in FNE(III).

5.2. The potential function

We define a potential function which will be used in the next subsection and Section 7. Unlike the frequently used single valued potential functions, our potential function is vector valued. Let $P(\cdot)$ be the potential function, $\pi = (B_1, B_2, \dots, B_m)$ a reasonable partition, and B_m the unique uncovered bin, then

$$P(\pi) = (s(B'_1), s(B'_2), \dots, s(B'_{m-1})),$$

where $\{B'_1, B'_2, \dots, B'_{m-1}\} = \{B_1, B_2, \dots, B_{m-1}\}$ and $s(B'_1) \leq s(B'_2) \leq \dots \leq s(B'_{m-1})$.

For any two vectors $v = (v_1, v_2, \dots, v_s)$, $w = (w_1, w_2, \dots, w_t)$, we say that v is lexicographically smaller than w , which is denoted by $v < w$, iff there exists an integer j_0 , $1 \leq j_0 \leq \min\{s, t\}$, such that $v_{j_0} < w_{j_0}$, and $\forall j < j_0$, $v_j = w_j$.

Accordingly, we can define an order $<$ between reasonable partitions. Let π^1, π^2 be two reasonable partitions, we say that $\pi^1 < \pi^2$ iff $P(\pi^1) < P(\pi^2)$. It is easy to check that this is a strictly partial order, that is, it satisfies irreflexivity ($\neg(\pi < \pi)$), asymmetry ($\pi^1 < \pi^2 \Rightarrow \neg(\pi^2 < \pi^1)$) and transitivity ($\pi^1 < \pi^2$ and $\pi^2 < \pi^3 \Rightarrow \pi^1 < \pi^3$). Therefore, the order is acyclic.

In the setting of FNE(I), where the migration can only occur from a covered bin to the uncovered one, the natural best response dynamic, in which an arbitrary unsatisfied agent moves to the uncovered bin until the partition is an equilibrium, will eventually always terminate, since each migration makes the previous partition smaller in the sense of $<$, and the number of reasonable partitions is finite for any fixed instance of SBC. Due to this argument, we get immediately that FNE(I) exists. A natural question is, is the best-response-dynamic algorithm polynomial? The answer is YES.

5.3. Main results

The basic ideas for the next two lemmas are from Yu and Zhang [31]. Starting from a rational partition $\pi = (B_1, B_2, \dots, B_m)$, where $s(B_1) \geq s(B_2) \geq \dots \geq s(B_m)$, and B_m is the unique uncovered bin, suppose the best-response-dynamic algorithm has K migrations before it ends. Let the partition after the t -th ($1 \leq t \leq K$) migration be $\pi^t = (B_1^t, B_2^t, \dots, B_m^t)$, and suppose that these bins have been re-indexed such that $s(B_1^t) \geq s(B_2^t) \geq \dots \geq s(B_m^t)$. Then:

Lemma 2. $s(B_m) < s(B_m^1) < s(B_m^2) < \dots < s(B_m^K)$, and $s(B_j) \geq s(B_j^1) \geq s(B_j^2) \geq \dots \geq s(B_j^K)$ for all $1 \leq j \leq m-1$.

Proof. Since in any step the minimal covered bin to which the migrating agent belongs becomes the next unique uncovered bin, the first part of the lemma is trivial. Because the total size of the new minimal covered bin is smaller than the one to which the migrating agent originally belongs, the second part is also easy. \square

Lemma 3. Suppose agent $i \in N$ migrates t_i times in the best-response-dynamic algorithm, then $t_i \leq m-1$. And hence $K \leq n(m-1)$.

Proof. Suppose the bin that agent i moves in for the j -th time ranks r_j in the derived partition, $1 \leq j \leq t_i$. For any fixed j , $j \geq 2$, let π^x, π^y and π^z be the partition after the $j-1$ -th move of agent i , the partition before the j -th move of agent i , and

the partition after the j -th move of agent i , respectively. It is trivial that $x < y < z$. We also have $i \in B_{r_{j-1}}^x, i \in B_{r_j}^z, s(B_{r_j}^z) = s(B_m^{z-1}) + a_i$ and $s(B_{r_{j-1}}^x) = s(B_m^{x-1}) + a_i$. Due to the first part of Lemma 2, we have $s(B_m^{z-1}) > s(B_m^{x-1})$ and therefore $s(B_{r_j}^z) > s(B_{r_{j-1}}^x)$. Due to the second part of Lemma 2, we have $s(B_{r_{j-1}}^z) \leq s(B_{r_{j-1}}^x)$. Therefore, $s(B_{r_j}^z) > s(B_{r_{j-1}}^z)$, and consequently $r_j < r_{j-1}$. This completes the proof. \square

Theorem 2. (a) $FNE(I)$ always exists, and computing an $FNE(I)$ can be done in $O(n^2)$ time;

(b) $PoS^{FNE(I)} = 1$;

(c) There exists an AFPTAS for approximating the best $FNE(I)$.

Proof. Since $m < n$, and a rational partition is trivial to calculate (say, by FFD), in $O(n)$ time, (a) is straightforward. (b) and (c) are valid for the same reasons as in Theorem 1. \square

Theorem 3. $PoA^{FNE(I)} = PoA^{FNE(II)} = PoA^{FNE(III)} = 0.5$.

Proof. The minimal covered property guarantees that none of the three equilibria has a PoA less than 0.5. To prove that they are all exactly 0.5, we only have to show that $PoA^{FNE(III)} = 0.5$, since $FNE(III) \subseteq FNE(II) \subseteq FNE(I)$.

We construct an instance as follows. There are $6n$ items in total: $2n$ large items with $a_1 = a_2 = \dots = a_{2n} = 2n - 2$, $4n$ small ones with $a_{2n+1} = a_{2n+2} = \dots = a_{6n} = 1$, and the volume of the bins is $b = 2n$. It is easy to check that the partition that each bin contains either two large items or $2n$ small ones is an $FNE(III)$. And the partition that each bin contains one large item and two small ones is optimal. Since $(n + 2)/(2n) \rightarrow 0.5$ as $n \rightarrow \infty$, we complete the whole proof. \square

Lemma 4. For any rational partition π , there exists a $\pi' \in FNE(III)$, such that $p(\pi') \geq p(\pi)$.

Proof. If π is not in $FNE(III)$, then there exists an agent $j \in N$ and a subset C_j of B_i ($1 \leq i \leq m, B_i \neq B_{j,\pi}$) such that $b \leq a_j + s(B_i \setminus C_j) < s(B_{j,\pi})$. We can construct a new rational partition $\pi' < \pi$, without decreasing the social welfare, in at most five steps as follows.

1. Move item j into B_i ;
2. If $C_j \neq \emptyset$, move the items of C_j into $B_{j,\pi}$;
3. If $(B_{j,\pi} \setminus \{j\} \cup C_j)$ is non-minimal covered, take out some subset D_j of items such that it is minimal covered;
4. If all the bins are covered, open a new bin with items in D_j , else put the items in D_j into the unique uncovered bin, which must be B_m ;
5. If $B_m \cup D_j$ is non-minimal covered, take out a subset E_j of items such that it is minimal covered, and open a new bin with items in E_j .

It is not hard to check that $\pi' < \pi$ and $p(\pi') \geq p(\pi)$. This completes the proof. \square

Theorem 4. $PoS^{FNE(II)} = PoS^{FNE(III)} = 1$.

Proof. Lemma 4 tells us that $PoS^{FNE(III)} = 1$. Since $FNE(III)$ is stronger than $FNE(II)$, the theorem is given. \square

Theorem 5. Computing an arbitrary $FNE(II)$ is NP-hard, and so is computing an arbitrary $FNE(III)$.

Proof. We prove this by reduction from the partition problem. Given any instance of the partition problem: e_1, e_2, \dots, e_n , we construct an instance of SBC as follows.

There are $n+2$ items in total: n small ones $a_1 = 2e_1, a_2 = 2e_2, \dots, a_n = 2e_n$, and 2 large ones $a_{n+1} = a_{n+2} = \sum_{j=1}^n e_j + 1$, and the volume of the bins is $b = \sum_{j=1}^n 2e_j + 1$. $\forall \pi \in FNE(II)$, it suffices to show that π has a bin that is exactly covered iff the answer to the partition problem is yes.

The necessity part is trivial. We show the sufficiency part. Let C be a set of items, and $\sum_{j \in C} e_j = (1/2) \sum_{j=1}^n e_j$. Since the total size of all the items is exactly $2b$, we know that π has exactly two bins. Suppose that $\pi = \{B_1, B_2\}$, and B_1 is minimal covered. There are two possibilities: **case 1.** B_1 has two large items; **case 2.** B_1 has one large item and a set of small ones. In case 1, B_2 will be exactly the set of small items, and it is uncovered. Therefore, item $n + 1$ will be strictly better off if he moves to B_2 and pushes out the items in C . This contradicts the fact that $\pi \in FNE(II)$. Therefore, this case will not occur.

In case 2, suppose B_1 is made up of item $n + 1$ and a set D of small items. We claim that B_1 is exactly covered, which will complete the whole proof. If not, we will have B_2 is uncovered and $\sum_{j \notin D} e_j < (1/2) \sum_{j=1}^n e_j$. Then $\sum_{j \in D} e_j \geq (1/2) \sum_{j=1}^n e_j + 1$. Therefore $s(B_1) = a_{n+1} + \sum_{j \in D} a_j \geq b + 3 > b + 2$, and item $n + 1$ will be strictly better off if it moves to B_2 and pushes out all the items other than item $n + 2$, because $a_{n+1} + a_{n+2} = b + 2$. This is a contradiction. \square

6. Strong Nash equilibrium

A partition $\pi = (B_1, B_2, \dots, B_m)$ is an SNE iff no group of agents can become all strictly better off by forming a new bin. That is, there is no subset $B \subseteq N$ such that $s(B) \geq b$ and $s(B) < s(B_{j,\pi})$ for all $j \in B$. It is trivial to see that SNE is stronger than NE and $FNE(III)$. Usually, SNE is too strong to exist. For SBC, however, it always exists and can be clearly characterized.

For any set of items E and its subset F , if $s(F) = \min\{s(G) : G \subseteq E, s(G) \geq b\}$, we say that F is a minimum subset w.r.t. (E, b) . The next lemma characterizes the SNE of SBC.

Lemma 5. Suppose $\pi = (B_1, B_2, \dots, B_m)$ is a reasonable partition, B_m is the unique uncovered bin, and $s(B_1) \leq s(B_2) \leq \dots \leq s(B_{m-1})$. Then π is an SNE iff B_k is a minimum subset w.r.t. $(\cup_{i=k}^m B_i, b)$ for all $1 \leq k \leq m-1$.

Proof (Sufficiency). Suppose that π is not an SNE, then there is a subset $B \subseteq N$ such that $s(B) \geq b$ and $s(B) < s(B_{j,\pi})$ for all $j \in B$. Let k_0 be the smallest index such that $B \cap B_{k_0} \neq \emptyset$, and $j_0 \in B \cap B_{k_0}$. Then $B \subseteq \cup_{i=k_0}^m B_i$. By hypothesis, $s(B_{k_0}) = \min\{s(G) : G \subseteq \cup_{i=k_0}^m B_i, s(G) \geq b\}$. Therefore $s(B_{k_0}) \leq s(B)$. This contradicts $s(B) < s(B_{k_0})$.

(necessity) Suppose that B_{k_0} is not a minimum subset w.r.t. $(\cup_{i=k_0}^m B_i, b)$, $1 \leq k_0 \leq m-1$, that is $s(B_{k_0}) > \min\{s(G) : G \subseteq \cup_{i=k_0}^m B_i, s(G) \geq b\}$. Let B be a minimum subset w.r.t. $(\cup_{i=k_0}^m B_i, b)$, then $b \leq s(B) < s(B_{k_0})$. Since $B \subseteq \cup_{i=k_0}^m B_i$, we know $s(B_{j,\pi}) \geq s(B_{k_0}) > s(B)$ for all $j \in B$. Therefore, $s(B) < s(B_{j,\pi})$ for all $j \in B$. A contradiction with $\pi \in \text{SNE}$. \square

Theorem 6. SNE exists, and to compute an arbitrary SNE is weakly NP-hard.

Proof. We only need to show the second part. We prove it by reduction from the partition problem. Let e_1, e_2, \dots, e_n be the input of the partition problem, and an instance of SBC is constructed simply as $a_j = e_j$, $1 \leq j \leq n$, and $b = (1/2) \sum_{j=1}^n e_j$. Let π be an SNE of SBC. It is straightforward that π has two exactly covered bins iff the answer to the partition problem is yes.

To prove that the problem is weakly NP-hard, it suffices to show that a pseudo-polynomial time algorithm is admitted. This is obvious since calculating $s(F) = \min\{s(G) : G \subseteq E, s(G) \geq b\}$ can be done in pseudo-polynomial time simply by recursively calling the algorithm to the subset sum problem. \square

The next two corollaries are both obvious.

Corollary 1. An arbitrary SNE, which is also FNE(III) and FNE(II), can be computed in $O(n \sum_{j=1}^n e_j)$ time.

Corollary 2. If b is a constant, then all the equilibria discussed in our paper can be computed in $O(n^2)$ time.

Theorem 7. $\text{PoA}^{\text{SNE}} = \text{PoS}^{\text{SNE}} = 0.5$.

Proof. For any $\pi = (B_1, B_2, \dots, B_m) \in \text{SNE}$, where B_m is the unique uncovered bin, we have $s(B_i) < 2b$ for all $1 \leq i \leq m-1$, then $\text{PoA}^{\text{SNE}} \geq 0.5$. It suffices to show that $\text{PoS}^{\text{SNE}} \leq 0.5$.

For any n , we construct an instance as follows. There are $4n$ items in total: $2n$ large items $a_1 = a_2 = \dots = a_{2n} = 2n-1$, and $2n$ small ones $a_{2n+1} = a_{2n+2} = \dots = a_{4n} = 2$, and $b = 2n$. Then all SNEs are made up of two small bins, each of which contains n small items, and n large bins, each of which contains two large items. And all of them have a social welfare of $n+2$. However, if we pair each large item with a small one, we will arrive at a social welfare of $2n$. Since $(n+2)/(2n) \rightarrow 0.5$ ($n \rightarrow \infty$), we get the whole theorem. \square

7. Modified strong Nash equilibrium

In the definition of FNE(I), further allowing the simultaneous migration of a group of agents from the same minimal covered bin into the unique uncovered bin gives the concept of modified strong Nash equilibrium (M-SNE). It is easy to see that M-SNE is stronger than FNE(I), but weaker than SNE. The exact definition is given as follows. Suppose that $\pi = (B_1, B_2, \dots, B_m)$ is a reasonable partition, and B_m is the unique uncovered bin.

Definition 4. π is said to be an M-SNE, iff it is rational, and for any $1 \leq i \leq m-1$, there is no subset $E \subseteq B_i$, such that $s(B_i \setminus E) > s(B_m)$ and $B_m \cup E$ is minimal covered.

Lemma 6. For any rational partition π , there exists a $\pi' \in \text{M-SNE}$, such that $p(\pi') \geq p(\pi)$.

Proof. If π is not an M-SNE, we can construct a partition π' from π by letting an unsatisfied group of agents migrating into the unique uncovered bin. It is easy to see that $\pi' < \pi$. \square

Theorem 8. $\text{PoA}^{\text{M-SNE}} = 0.5$, $\text{PoS}^{\text{M-SNE}} = 1$.

Proof. Lemma 6 guarantees that $\text{PoS}^{\text{M-SNE}} = 1$. $\text{PoA}^{\text{SNE}} = 0.5$ tells us that $\text{PoA}^{\text{M-SNE}} \leq 0.5$. Since for each bin B in a rational partition, $s(B) < 2b$, we get $\text{PoA}^{\text{M-SNE}} \geq 0.5$. Therefore, $\text{PoA}^{\text{M-SNE}} = 0.5$. \square

Theorem 9. To compute an arbitrary M-SNE is NP-hard.

Proof. Again, we prove by reduction from the partition problem. Given any instance of the partition problem: e_1, e_2, \dots, e_n , we construct an instance of SBC as follows.

There are $n+1$ items in total: $a_j = 2e_j$, for all $1 \leq j \leq n$, $a_{n+1} = \sum_{j=1}^n e_j - 1$, and $b = \sum_{j=1}^n 2e_j - 1$. Let $\pi = (B_1, B_2)$ be an M-SNE. Obviously, exactly one of the two bins is covered. Suppose B_1 is covered. We prove that the answer to the partition problem is yes iff B_1 is exactly covered.

The sufficiency part is trivial, so we only need to show the necessity part. Suppose $\sum_{j \in C} e_j = (1/2) \sum_{j=1}^n e_j$. There are two possibilities: $n+1 \in B_1$ and $n+1 \notin B_1$. In the latter case, we have $B_1 = \{1, 2, \dots, n\}$. Therefore $s(B_1) = \sum_{j=1}^n 2e_j = b+1 > b$, and the items in C will benefit if they move into B_2 . Hence this case will not occur. In the former case, if $s(B_1) > b$, we will have $s(B_1) \geq b+2$, and the items in $B_1 \setminus \{n+1\}$ will benefit if they move into B_2 , a contradiction. \square

Since M-SNE is defined analogously to FNE(I), it should actually be written as M-SNE(I), and we can also define M-SNE(II) and M-SNE(III), analogously to FNE(II) and FNE(III), respectively. It is not hard to show that all the results for M-SNE, i.e. NP-hardness, $\text{PoA} = 0.5$ and $\text{PoS} = 1$, also hold for M-SNE(II) and M-SNE(III).

8. Further discussions

In this paper, we discussed the selfish bin covering problem. In order to see how decentralized decision making affects the social welfare, various PoAs and PoSs are analyzed. We mainly considered six equilibrium concepts, whose relations are summarized as follows.

$$\begin{array}{ccccc}
 FNE(III) & \Rightarrow & FNE(II) & \Rightarrow & FNE(I) \\
 \uparrow & & & & \uparrow \\
 SNE & & \Rightarrow & & M - SNE \\
 \downarrow & & & & \\
 NE & & & &
 \end{array}$$

It is easy to show that the above relation graph is complete, i.e., all the other relations that are not expressed in this graph are false.

A frequently asked question is, among so many equilibrium concepts, which is the most reasonable one? Our view is that this depends on more specifics of the game environment which are not explicitly expressed. By adding different assumptions or different specifics, we would naturally get different equilibrium concepts. It is meaningless to discuss which equilibrium concept is good and which is bad, because different concepts apply to different situations. For example, it is not so appropriate to say that strong Nash equilibrium is better than Nash equilibrium. In the scenario where Nash equilibrium applies, group deviation is simply not allowed.

In the field of mechanism design, however, it is meaningful to compare different equilibrium concepts, because various restrictions are decision variables rather than fixed parameters. It is also meaningful to study how a weak equilibrium concept approximates a stronger one, i.e. how much incentive is given in any weak equilibrium to the corresponding agents by deviation that is allowed only in the scenario for the stronger equilibrium. See [5,14] for deep discussions for NE and SNE on scheduling games.

If we consider asymptotic PoAs and asymptotic PoSs, i.e. in the definitions we change \inf as \liminf and let $OPT(G) \rightarrow \infty$, then all the results remain unchanged. This can be interpreted as that these measures are pretty stable with regard to the size of the problems.

The algorithm FFD-LPT in Section 4 is not a best-response-dynamic algorithm. An open problem is that starting from any rational partition, does any best-response-dynamic algorithm, which always converges to an NE, end in polynomial time, as we showed for FNE(I)? We know this is impossible for FNE(II) or FNE(III), as they are both NP-hard to compute. It is also interesting to study whether SNE and M-SNE can be arrived at in a polynomial number of migrations.

The following are some interesting directions for further research.

1. To consider other payoff allocation rules, e.g. the membership rule, i.e. the payoff of a covered bin is shared equally among its members, allocation rule proportional to varieties of power indices, etc.;
2. To consider the *egalitarian* social welfare function, i.e. the social welfare is determined by the lowest payoff among all the players. The corresponding combinatorial optimization problem can be easily shown to be strongly NP-hard, by reduction from the 3-partition problem;
3. To take a hybrid social welfare of the utilitarian rule and the egalitarian rule, either aggregating them into one function using the weighting way, or discussing them lexicographically, or any other method dealing with bi-criteria problems;
4. To extend the model to heterogeneous bin payoffs, i.e. the payoff gained by every covered bin is a function of its total size. Notice that this is closely related to the congestion game with a parallel network;
5. To extend the model to heterogeneous bins: variable sized bins, variable payoff bins. Notice that these models are closely related with the selfish load balancing, i.e. scheduling games, on uniformly related machines;
6. To extend the model to higher dimensional cases, i.e. items and bins are characterized by vectors of a fixed dimension;
7. To discuss the ordinal model, i.e. how much a covered bin gains is determined by its rank among all the covered bins. Notice that externality occurs in this model;
8. To discuss incompatible agent families, i.e. agents from the same family cannot be packed into the same bin;
9. To discuss a hybrid model of centralization and decentralization, i.e. each agent may control more than one item;
10. To discuss online coalition formations, i.e. agents arrive one by one, and each agent decides either to join an old bin, or to open a new bin, e.g. the procedure proposed by Maskin [22].

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